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Wave reflection from a gently sloping beach

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The linear reflection of an obliquely incident gravity wave of frequency ω from a gently sloping beach of shoreline slope σ and characteristic length l is determined for $\sigma \ll 1 \ll \omega^2 l/g$. An asymptotic ($\sigma \downarrow 0$), inviscid approximation that is uniformly valid in the shallow-water domain is matched to Keller's (1958) geometrical-optics approximation for non-shallow water. An exact solution is obtained for the profile $h = \sigma l[1-\exp(-x/l)]$ in the shallow-water domain and used to test the asymptotic approximation. The absence of viscosity implies perfect reflection. A model that incorporates both small viscosity and small capillarity predicts a fixed contact line and the reflection coefficient $|R| = \exp[-\pi\sigma^{-2}g^{-1}(2\nu\omega^3)^{\frac{1}{2}}]$, where ν is the kinematic viscosity. These predictions are in qualitative agreement with the experimental results of Mahony & Pritchard (1980).

1. Introduction

I consider here the linear reflection of a gravity wave from a gently sloping beach of depth

$$h(x) = \sigma l H(x/l) \sim \begin{matrix} \sigma x \ (x/l \downarrow 0) \\ h_\infty \ (x/l \uparrow \infty) \end{matrix}, \tag{1.1}$$

where σ is the shoreline ($x = 0$) slope, $l = O(h_\infty/\sigma)$ is a characteristic length of the beach, H is a smooth, monotonically increasing function, and h_∞ is the offshore depth. The characteristic length for a wave of frequency ω ranges from $(h_\infty/K)^{\frac{1}{2}}$ to $1/K$, where

$$K \equiv \omega^2/g, \tag{1.2}$$

and the parameters of the inviscid problem are σ and Kh_∞. The free-surface displacement has the form

$$\zeta(x,y,t) = a_0 Z(x)\cos(\omega t - k_y y) \quad (0 \leqslant x < \infty, \quad -\infty < y < \infty), \tag{1.3}$$

where a_0 is the shoreline ($x = 0$) amplitude, k_y is the longshore wavenumber, and $Z(x)$ is to be determined subject to the boundary conditions ($\zeta_{\max} \equiv a_0$ and mass flux = 0)

$$Z = 1, \quad hZ' = 0 \quad (x = 0). \tag{1.4a, b}$$

The absence of dissipation implies perfect reflection, by virtue of which

$$Z \sim A\cos[(k_\infty^2 - k_y^2)^{\frac{1}{2}}x + \psi] \quad (k_\infty x \uparrow \infty), \tag{1.5}$$

where k_∞ is determined by the gravity-wave dispersion relation

$$k_\infty \tanh(k_\infty h_\infty) = K, \tag{1.6}$$

and the offshore amplitude $a_\infty \equiv Aa_0$ and phase shift ψ are to be obtained as part of the solution (although in practice a_∞ is specified and a_0 is inferred from $A = a_\infty/a_0$).

The earliest solutions of this linear, inviscid reflection problem (see Stoker 1957, §5.1) are for uniform slope with $\sigma = \tan(\pi/2n), n = 1, 2, \ldots$, but these solutions are cumbersome for the most important (for oceanography) case of small slope. Matched asymptotic approximations for normal incidence and $\sigma \ll 1$ have been determined for uniform slope by Friedrichs (1948) and for non-uniform slope by Keller (1961), who matched his (1958) geometrical-optics approximation for $Kl \gg 1$ to the shallow-water approximation (Lamb 1932, §186)

$$Z = J_0[2(Kx/\sigma)^{\frac{1}{2}}] \quad (h \sim \sigma x \ll 1/K), \tag{1.7}$$

where J_0 is a Bessel function. Carrier & Greenspan (1958) have solved the nonlinear problem for non-breaking reflection from a uniform slope and shown that $Ka_0 < \sigma^2$ is necessary for a smooth solution. Their solution reduces to (1.7) for $Ka_0 \ll \sigma^2$.

The approximation (1.7) is often adopted in the investigation of edge waves (see Guza 1985 for a review) and other shore processes on the hypothesis that the disturbances associated with these processes are essentially confined to the domain of uniform slope; however, this hypothesis fails in some problems, and, in any event, its testing requires the development of approximations that incorporate non-uniform slope. Towards that end, I develop (in §2) an approximation that is uniformly valid in $x = O(l)$ for $\sigma \ll 1 \ll Kl \ll 1/\sigma$ and (in §4) match that approximation to a geometrical-optics approximation for arbitrary Kh_∞. In §3, I obtain an exact solution of the shallow-water equations for an exponential profile through an extension of Ball's (1967) solution for edge waves. This exact solution provides a test of the asymptotic approximations of §2.

Viscosity is almost always significant in laboratory experiments (although it may be negligible for non-breaking waves of geophysical scale). In §5, I consider its effects on normally incident waves on the assumption of no slip at the bottom, $K\delta_* \ll \sigma^2$, and $\lambda = O(\delta_*/\sigma)$, where $\delta_* \equiv (\nu/2\omega)^{\frac{1}{2}}$ is a viscous lengthscale (ν is the kinematic viscosity), and λ is the capillary length (2.8 mm for clean water). The corresponding extension of the shallow-water equation (Miles 1990) predicts total absorption (zero reflection) of the incident wave if capillarity is neglected ($\lambda = 0$). If $0 < \lambda = O(\delta_*/\sigma)$ capillarity is significant only in an inner approximation, which may be matched to an outer, boundary-layer approximation (which assumes $h \gg \delta_*$ and manifestly fails as $h \downarrow 0$). This matching determines the reflection coefficient. If h is given by (1.1) for $x \lessgtr x_1$ (so that $h_\infty = \sigma x_1$), as in a typical laboratory experiment, the reflection coefficient referred to $x = x_1$ is given by

$$|R_1| = \exp[-2\pi(K\delta_*/\sigma^2)] \tag{1.8}$$

for a clean free surface. The exponent is doubled for a fully contaminated surface. (Note that dissipation implies an exponential increase/decrease of the incident/reflected wave as $x \uparrow \infty$, in consequence of which $|R|$ is intrinsically sensitive to the location of the reference station.) It should be emphasized that the assumption of no slip at the bottom implies a fixed contact line. This prediction is supported by Mahony & Pritchard's (1980) laboratory experiments, but not by oceanographic observation.

Mahony & Pritchard (1980) measured $|R_1| = 0.114$ for $\sigma = 0.09$ and $2\pi/\omega = 0.70$ s, for which (1.8) yields $|R_1| = 0.22$ on the assumption of a clean surface or 0.048 for a fully contaminated surface. Guza & Bowen (1976) report almost perfect reflection for $\sigma = 0.1$ and $2\pi/\omega = 2.4$–3.4 s, for which (1.8) predicts $|R_1| = 0.90$–0.82; the difference $1 - |R_1|$ is within the accuracy of their measurements, but their contact line was moving (Guza, private communication).

2. Shallow-water domain

The assumptions of incompressible, irrotational fluid motion with

$$Ka_0 \ll \sigma^2, \quad Kh \ll 1, \tag{2.1a, b}$$

lead to the shallow-water equation (Lamb 1932, §193)

$$g\nabla\cdot(h\nabla\zeta) = \partial^2\zeta/\partial t^2. \tag{2.2}$$

Substituting (1.3) into (2.2) and invoking $\omega^2/g \equiv K$, we obtain the Sturm–Liouville equation

$$(hZ')' + (K - k_y^2 h) Z = 0. \tag{2.3}$$

We seek the solution of (2.3), subject to (1.4), for

$$\epsilon \equiv \sigma/Kl \ll 1. \tag{2.4}$$

2.1. *Asymptotic solution*

Guided by (1.7) and by Erdélyi's (1955, §4.1) treatment of Liouville's problem, we posit

$$Z(x) = [h(K-k_y^2 h)]^{-\frac{1}{4}}(\tfrac{1}{2}\sigma\chi)^{\frac{1}{2}} f(\chi), \tag{2.5a}$$

where
$$\chi = \int_0^x \left(\frac{K-k_y^2 h}{h}\right)^{\frac{1}{2}} \mathrm{d}x = \epsilon^{-\frac{1}{2}} \int_0^\xi (H^{-1}-\beta)^{\frac{1}{2}} \mathrm{d}\xi \quad (\xi \equiv x/l), \tag{2.5b}$$

and
$$\beta \equiv \sigma k_y^2 l/K = \epsilon (k_y l)^2 \tag{2.6}$$

is a measure of obliquity. Transforming (2.3) and (1.4), we obtain

$$f'' + \chi^{-1} f' + f = rf \tag{2.7}$$

and
$$f = 1, \quad \chi f' = 0 \quad (\chi = 0), \tag{2.8a, b}$$

where
$$r = \tfrac{1}{4}\left\{\frac{(K-2k_y^2 h)h''}{(K-k_y^2 h)^2} - \frac{(K^2+4k_y^2 h^2)h'^2}{4(K-k_y^2 h)^3 h} + \chi^{-2}\right\} \equiv r(\chi), \tag{2.9}$$

$f' \equiv \mathrm{d}f/\mathrm{d}\chi$, and $h' \equiv \mathrm{d}h/\mathrm{d}x$. We remark that $r(\chi)$ is regular and has the limiting values

$$r(0) = \tfrac{1}{6}\epsilon(H_0'' - \beta), \quad r \sim \tfrac{1}{4}\chi^{-2} \; (\epsilon\chi^2 \uparrow \infty). \tag{2.10a, b}$$

A first approximation to the solution of (2.7) and (2.8) is given by

$$f = J_0(\chi) + O(\epsilon). \tag{2.11}$$

It follows that
$$Z(x) = [h(K-k_y^2 h)]^{-\frac{1}{4}}(\tfrac{1}{2}\sigma\chi)^{\frac{1}{2}} J_0(\chi) + O(\epsilon), \tag{2.12}$$

where χ is given by (2.5b), provides a uniformly valid approximation to the solution of (1.4) and (2.3). Letting $\sigma x \downarrow 0$, we recover the inner approximation (1.7). Letting $\chi \uparrow \infty$, we obtain the outer approximation

$$Z(x) \sim (\sigma/\pi)^{\frac{1}{2}}[h(K-k_y^2 h)]^{-\frac{1}{4}}[\cos(\chi - \tfrac{1}{4}\pi) + O(\chi^{-1})] \quad (\chi \uparrow \infty). \tag{2.13}$$

Letting $h \sim h_\infty$ in (2.13) with $Kh_\infty \ll 1$, comparing with (1.5), and invoking

$$k_y^2 h_\infty/K = (k_y/k_\infty)^2 = \sin^2\theta_1 \quad (Kh_\infty \ll 1), \tag{2.14}$$

where $k_\infty = \omega/(gh_\infty)^{\frac{1}{2}}$ is the offshore wavenumber and θ_1 is the angle of incidence, we obtain

$$A = (\sigma/\pi)^{\frac{1}{2}}(Kh_\infty)^{-\frac{1}{4}}(\sec\theta_1)^{\frac{1}{2}} = \sigma^{\frac{1}{2}}(\pi k_\infty h_\infty \cos\theta_1)^{-\frac{1}{2}} \tag{2.15a}$$

and $$\psi = \int_0^\infty [(Kh^{-1} - k_y^2)^{\frac{1}{2}} - k_\infty \cos\theta_1]\,dx - \tfrac{1}{4}\pi. \tag{2.15b}$$

2.2. *Higher approximations*

To construct higher approximations, we regard (2.7) as an inhomogeneous Bessel equation, solve by variation of parameters, and invoke (2.8) to obtain the Volterra integral equation

$$f(\chi) = J_0(\chi) + \int_0^\chi G(\chi, \hat{\chi})\, r(\hat{\chi}) f(\hat{\chi})\, d\hat{\chi}, \tag{2.16a}$$

where $$G(\chi, \hat{\chi}) = \tfrac{1}{2}\pi\hat{\chi}[J_0(\hat{\chi})\, Y_0(\chi) - J_0(\chi)\, Y_0(\hat{\chi})]. \tag{2.16b}$$

The solution of (2.16) may be obtained by iteration. Substituting the first approximation $f(\hat{\chi}) = J_0(\hat{\chi})$ into (2.16a) and integrating by parts, we obtain the second approximation

$$f(\chi) = J_0(\chi) + \int_0^\chi G(\chi, \hat{\chi})\, J_0(\hat{\chi})\, r(\hat{\chi})\, d\hat{\chi} + O(\epsilon^2) \tag{2.17a}$$

$$= J_0(\chi) + \epsilon r(\chi) \int_0^\chi G(\chi, \hat{\chi})\, J_0(\hat{\chi})\, d\hat{\chi} + O(\epsilon^2) \tag{2.17b}$$

$$= J_0(\chi) + \tfrac{1}{2} r(\chi)\, J_1(\chi) + O(\epsilon^2) = J_0[(1-r)^{\frac{1}{2}}\chi] + O(\epsilon^2) \tag{2.17c}$$

((2.17c) may be derived heuristically by regarding r as slowly varying in (2.7)). The outer approximation (2.13) remains unchanged in these higher approximations and is valid to any algebraic order in ϵ (but it may exhibit an exponentially small error as $\epsilon \downarrow 0$; see §3).

3. Exponential profile

The shallow-water equation (2.3) admits exact solutions for (cf. Ball 1967)

$$h = \sigma l(1 - e^{-x/l}) \quad (H = 1 - e^{-\xi}, \quad h_\infty = \sigma l). \tag{3.1}$$

Adopting H as the independent variable, we obtain

$$Z = \mathrm{Re}\{e^{i\tau\xi} F(\tfrac{1}{2} + \rho - i\tau, \tfrac{1}{2} - \rho - i\tau; 1; 1 - e^{-\xi})\} \tag{3.2a}$$

$$= \mathrm{Re}\{\mathscr{A}\, e^{i\tau\xi} F(\tfrac{1}{2} + \rho - i\tau, \tfrac{1}{2} - \rho - i\tau; 1 - 2i\tau; e^{-\xi})\}, \tag{3.2b}$$

where Re signifies the real part of, F is Gauss's hypergeometric function,

$$\rho \equiv (\tfrac{1}{4} + \epsilon^{-1}\sin^2\theta_1)^{\frac{1}{2}}, \quad \tau \equiv \epsilon^{-\frac{1}{2}}\cos\theta_1, \tag{3.3a, b}$$

$$\mathscr{A} = \frac{2\Gamma(2i\tau)}{\Gamma(\frac{1}{2} + \rho + i\tau)\,\Gamma(\frac{1}{2} - \rho + i\tau)}, \tag{3.4}$$

and θ_1 is the angle of incidence given by (2.14) with $h_\infty = \sigma l$ therein.

Letting $\xi \uparrow \infty$ in (3.2b), comparing the result with (1.5), and invoking $\Gamma(z) \times \Gamma(1-z) = \pi \operatorname{cosec} \pi z$ and $|\Gamma(iy)| = (\pi/y \sinh \pi y)^{\frac{1}{2}}$, we obtain

$$A = |\mathscr{A}| = (\pi\tau)^{-\frac{1}{2}}(\cos^2\pi\rho \coth\pi\tau + \sin^2\pi\rho \tanh\pi\tau)^{\frac{1}{2}} \tag{3.5a}$$

and $$\psi = \arg\mathscr{A} = \arg\Gamma(2i\tau) - 2\arg\Gamma(\tfrac{1}{2} + \rho + i\tau) + \tan^{-1}(\tan\pi\rho \tanh\pi\tau). \tag{3.5b}$$

The approximation implied by (2.15a) with $h_\infty = \sigma l$ therein is $A = (\pi\tau)^{-\frac{1}{2}}$, which is exponentially close to (3.5a) in the limit $\epsilon \downarrow 0$ and differs therefrom by less than 0.01% for $\epsilon \leqslant \frac{1}{2}$.

It does not appear possible to reduce (3.5b) to elementary transcendents, but asymptotic ($\epsilon \downarrow 0$) approximations may be obtained from the asymptotic expansion of the gamma function. For normal incidence ($\rho = \frac{1}{2}, \tau = \epsilon^{-\frac{1}{2}}$),

$$\psi \sim \epsilon^{-\frac{1}{2}} \ln 4 - \tfrac{1}{4}\pi + \tfrac{1}{8}\epsilon^{\frac{1}{2}} + O(\epsilon^{\frac{3}{2}}). \tag{3.6}$$

The corresponding approximation implied by (2.15b) is

$$\psi = \epsilon^{-\frac{1}{2}} \int_0^\infty [(1-\mathrm{e}^{-\xi})^{-\frac{1}{2}} - 1]\,\mathrm{d}\xi - \tfrac{1}{4}\pi = \epsilon^{-\frac{1}{2}} \ln 4 - \tfrac{1}{4}\pi, \tag{3.7}$$

which differs from (3.6) by less than 7.5% for $\epsilon < \frac{1}{2}$.

4. Transition to non-shallow water

We now suppose that $Kh \ll 1$ is satisfied out to a depth for which (2.13) is a valid approximation, but that h continues to increase (with x) to non-shallow values. It then follows from Keller's (1958) geometrical-optics approximation that $Z(x)$ has the form

$$Z = A(x) \cos[\Psi(x)] + O(1/Kl), \tag{4.1}$$

where A and Ψ satisfy

$$\Psi'^2 + k_y^2 = k^2, \quad [A^2 \operatorname{sech}^2 kh(\sinh^2 kh + Kh)\,\Psi']' = 0, \tag{4.2a, b}$$

and

$$k \tanh kh = K. \tag{4.3}$$

Integrating (4.2), matching (4.1) to (2.13) to determine the constants of integration, and eliminating the hyperbolic functions with the aid of (4.3), we obtain

$$\Psi = \int_0^x (k^2 - k_y^2)^{\frac{1}{2}}\,\mathrm{d}x - \tfrac{1}{4}\pi, \tag{4.4a}$$

$$A = (2\sigma\pi)^{\frac{1}{2}} k (k^2 - k_y^2)^{-\frac{1}{4}} [K + h(k^2 - K^2)]^{-\frac{1}{2}}. \tag{4.4b}$$

Letting $k_y = 0$, we recover Keller's (1961) results.

Letting $h \to h_\infty \to \infty$ in (4.3) and (4.4), we obtain $k \to k_\infty \to K$ and

$$A \sim (2\sigma/\pi)^{\frac{1}{2}} (\sec\theta_{\mathrm{i}})^{\frac{1}{2}} \quad (Kh_\infty \uparrow \infty) \tag{4.5}$$

as the deep-water counterpart of the shallow-water approximation (2.15a).

5. Viscous and capillary effects

We now admit viscosity and capillarity in the two-dimensional, shallow-water domain. Assuming normal incidence and replacing (1.3) by

$$\zeta = a \operatorname{Re}\{Z(x)\,\mathrm{e}^{-\mathrm{i}\omega t}\}, \tag{5.1}$$

where a is an amplitude scale, Re implies the real part of, and Z is a dimensionless, complex amplitude, we find that (2.3) is replaced by (Miles 1990)

$$[p(Z - \lambda^2 Z'')']' + KZ = 0, \quad p = h - \delta \tanh(h/\delta), \tag{5.2a, b}$$

where

$$\delta \equiv (\nu/\omega)^{\frac{1}{2}} \mathrm{e}^{\frac{1}{4}\mathrm{i}\pi} \equiv (1+\mathrm{i})\,\delta_*, \tag{5.3}$$

$\delta_* = (\nu/2\omega)^{\frac{1}{2}}$ is a viscous lengthscale, and $\lambda \equiv (T/\rho g)^{\frac{1}{2}}$ is the capillary length (T is the surface tension). The corresponding boundary conditions (for $h \sim \sigma x$ as $x \downarrow 0$) are

$$Z = 0, \quad p(Z-\lambda^2 Z'')' = 0 \quad (x = 0). \tag{5.4a, b}$$

The preceding formulation is for a clean surface. If the surface is fully contaminated (inextensible) δ is replaced by 2δ in (5.2*b*).

5.1. *Inner and outer approximations*

We proceed on the assumptions that $|\alpha| \ll 1$ and $\gamma = O(1)$, where

$$\alpha = \frac{K\delta}{\sigma^2}, \quad \gamma \equiv \frac{\lambda\sigma}{\delta}. \tag{5.5a, b}$$

Capillarity then is negligible in $h \gg \delta_*$, and (5.2) admits an outer approximation of the form (cf. (2.12) with $k_y = 0$ therein)

$$Z = [K(h-\delta)]^{-\frac{1}{4}}(\tfrac{1}{2}\sigma\chi)^{\frac{1}{2}}[J_0(\chi)\cos\chi_0 - Y_0(\chi)\sin\chi_0], \tag{5.6a}$$

where
$$\chi = \int_{\delta/\sigma}^{x}\left(\frac{K}{h-\delta}\right)^{\frac{1}{2}} \mathrm{d}x, \tag{5.6b}$$

and, by hypothesis, $\chi_0 = O(\alpha)$.

Assuming that $h \approx \sigma x$ for $\sigma x = O(\delta)$ and introducing the inner variables

$$\xi = \frac{h}{\delta} = \frac{\sigma x}{\delta}, \quad Z = Z_i(\xi), \tag{5.7a, b}$$

we transform (5.2) to

$$[\varpi(Z_i - \gamma^2 Z_i'')']' + \alpha Z_i = 0, \quad \varpi = \xi - \tanh\xi. \tag{5.8a, b}$$

Integrating (5.8*a*) from 0 to ξ, so that (5.4*b*) is satisfied, dividing by ϖ, and integrating again, we obtain

$$Z_i - \gamma^2 Z_i'' = C_1 + \alpha F(\xi), \quad F(\xi) \equiv -\int_{\xi_1}^{\xi}\frac{\mathrm{d}\eta}{\varpi(\eta)}\int_0^{\eta} Z_i(\zeta)\,\mathrm{d}\zeta, \tag{5.9a, b}$$

where C_1 and ξ_1 are interdependent constants of integration. Integrating (5.9*a*) and invoking (5.4*a*) and $Z_i = O(1)$ for $\xi \to \infty$, we obtain

$$Z_i = C_1(1-\mathrm{e}^{-\xi/\gamma}) + \frac{\alpha}{2\gamma}\int_0^{\infty}[\mathrm{e}^{-|\xi-\eta|/\gamma} - \mathrm{e}^{-(\xi+\eta)/\gamma}]F(\eta)\,\mathrm{d}\eta. \tag{5.10}$$

The integral equation (5.10) may be solved by iteration, starting from the first approximation $C_1[1-\exp(-\xi/\gamma)]$, the substitution of which into (5.9*b*) yields

$$F = -C_1\int_{\xi_1}^{\xi}\left[\frac{\eta - \gamma(1-\mathrm{e}^{-\eta/\gamma})}{\varpi(\eta)}\right]\mathrm{d}\eta, \tag{5.11}$$

where, here and subsequently, error factors of $1+O(\alpha)$ are implicit. Letting $\xi \to \infty$, anticipating that $|\xi_1| \gg 1$ so that $\varpi(\xi) \sim \xi - 1$ holds throughout the range of integration in (5.11), and substituting the resulting approximation to F into (5.9*a*), we obtain

$$Z_i \sim C_1\left\{1 - \alpha\left[\xi - \xi_1 + (1-\gamma)\log\left(\frac{\xi-1}{\xi_1-1}\right)\right] + O\left[\frac{\alpha\gamma^2(1-\gamma)}{(\xi-1)^2}\right]\right\}. \tag{5.12}$$

Returning to the outer approximation (5.6) and letting $h \sim \sigma x$ and $\chi \to 0$, we obtain

$$\chi \sim 2\sigma^{-1}[K(h-\delta)]^{\frac{1}{2}} = 2[\alpha(\xi-1)]^{\frac{1}{2}} \tag{5.13a}$$

and
$$Z \sim [1-\alpha(\xi-1)]\cos\chi_0 - \pi^{-1}\sin\chi_0 \log[C^2\alpha(\xi-1)], \tag{5.13b}$$
where $C = 1.78\ldots$ is Euler's constant. We match (5.12) and (5.13*b*) by choosing
$$C_1 = \cos\chi_0, \quad \tan\chi_0 = \pi\alpha(1-\gamma), \quad \xi_1 + (1-\gamma)\log(\xi_1 - 1) = 1 + (1-\gamma)\log(1/C\alpha^2), \tag{5.14a–c}$$
wherein error factors of $1+O(\alpha^2)$ are implicit.

Finally, we let $\chi \to \infty$ in (5.6*a*) to obtain (cf. (2.13) with $k_y = 0$ therein)
$$Z \sim (\sigma/\pi)^{\frac{1}{2}}[K(h-\delta)]^{-\frac{1}{4}}\cos(\chi+\chi_0-\tfrac{1}{4}\pi), \quad \chi \sim k_\infty x + \chi_*, \tag{5.15a, b}$$
where
$$k_\infty = \left(\frac{K}{h_\infty - \delta}\right)^{\frac{1}{2}}, \quad \chi_* = \frac{-k_\infty \delta}{\sigma} + K^{\frac{1}{2}}\int_{\delta/\sigma}^{\infty}[(h-\delta)^{-\frac{1}{2}} - (h_\infty - \delta)^{-\frac{1}{2}}]\,dx. \tag{5.16a, b}$$
Combining (5.15) and (5.16), we place the result in the form
$$Z = \tfrac{1}{2}(\sigma/\pi)^{\frac{1}{2}}R_0^{-\frac{1}{2}}[K(h-\delta)]^{-\frac{1}{4}}(e^{-ik_\infty x} + R_0 e^{ik_\infty x})], \tag{5.17a}$$
where
$$R_0 = \exp[2i(\chi_0 + \chi_* - \tfrac{1}{4}\pi)] \tag{5.17b}$$
is the reflection coefficient referred to $x = 0$.

We are interested primarily in the magnitude of R_0. Letting
$$|R_0| \equiv e^{-\rho}, \tag{5.18}$$
and approximating $\tan\chi_0$ by χ_0 in (5.14*b*) (note that $\alpha\gamma$ is real), we obtain
$$\rho = 2\pi\,\mathrm{Im}\,\alpha = 2\pi\sigma^{-2}K\delta_*. \tag{5.19}$$

5.2. *Wedge profile*

The development in (5.15)–(5.17) assumes smooth $h(x)$, but in the typical wave tank
$$h = \frac{\sigma x}{h_1}\left(x \lessgtr x_1 \equiv \frac{h_1}{\sigma}\right). \tag{5.20}$$
and h' is discontinuous at $x = x_1$. The outer approximation (5.6) then reduces to
$$Z = J_0(\chi)\cos\chi_0 - Y_0(\chi)\sin\chi_0, \quad \chi = 2\sigma^{-1}[K(h-\delta)]^{\frac{1}{2}} \quad (x < x_1), \tag{5.21a, b}$$
while the solution in $x > x_1$ has the form
$$Z = \left(\frac{A_1}{1+R_1}\right)[e^{-ik_1(x-x_1)} + R_1 e^{ik_1(x-x_1)}], \quad k_1 = \left(\frac{K}{h_1-\delta}\right)^{\frac{1}{2}} \quad (x > x_1), \tag{5.22a, b}$$
where R_1 is the reflection coefficient referred to $x = x_1$ (note that $h_1 \equiv h_\infty$ and $k_1 \equiv k_\infty$). Requiring Z and Z' to be continuous at $x = x_1$, we obtain
$$A_1 = J_0(\chi_1)\cos\chi_0 - Y_0(\chi_1)\sin\chi_0, \quad R_1 = \exp\left\{2i\tan^{-1}\left[\frac{J_1(\chi_1)\cos\chi_0 - Y_1(\chi_1)\sin\chi_0}{J_0(\chi_1)\cos\chi_0 - Y_0(\chi_1)\sin\chi_0}\right]\right\}. \tag{5.23a, b}$$
Letting $\chi_1 \to \infty$ in (5.23*b*), we obtain
$$R_1 \sim \exp\{2i(\chi_0 + \chi_1 - \tfrac{1}{4}\pi)[1 + O(\chi_1^{-1})]\}. \tag{5.24}$$
Comparing (5.24) and (5.17*b*) and invoking (5.16*a*, *b*), $h_1 = h_\infty$ and $k_1 = k_\infty$, we obtain
$$R_1 = R_0\rho^{2ik_1x_1}, \quad |R_1| = e^{-\rho}[1 + O(\chi_1^{-1})]. \tag{5.25a, b}$$

Considering, for example, the experiments of Mahony & Pritchard (1980), for which $\omega = 9.06\ \mathrm{s}^{-1}$, $\sigma = 0.090$, $h_1 = 3$ cm, $g = 980\ \mathrm{cm/s^2}$, and $\lambda = 0.28$ cm (for clean water), we obtain $K = 0.084\ \text{cm}^{-1}$, $|\delta| = 0.033$ cm, $|\alpha| = 0.34$, $|\gamma| = 0.76$, $|\chi_1| = 11.2$, $\rho = 1.52$ and $|R_1| = 0.22$. The observed value was $|R_1| = 0.114$ ($\rho = 2.17$). If the surface were fully contaminated δ would be doubled, which would yield $\rho = 3.0$ and $|R_1| = 0.048$. Mahony & Pritchard state that 'the surface of the water was skimmed with a vacuum pump' before the start of each experiment, but this may not have been sufficient to avoid at least partial contamination before the completion of a particular measurement. Allowing for this possibility, we conclude that the agreement between the present calculation of $|R_1|$ and Mahony & Pritchard's observed value is within the uncertainties associated with the error factor $1+O(\alpha)$ in the calculation and the free-surface condition in the experiments.

Guza & Bowen (1976) observed almost perfect reflection for $\sigma = 0.1$ and $2\pi/\omega =$ 2.4–3.4 s ($|\alpha| = 0.043$–0.025), for which (5.26) predicts $|R_1| = 0.90$–0.82 for clean water. However, their estimate of complete reflection was based on the fit of the observed profile to that predicted by the inviscid theory and is as consistent with $|R_1| = 0.8$ as with $|R_1| = 1$ (Guza, private communication).

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